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The Hausdorff Paradox for General Group Actions

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A well-known criterion for the amenability of a locally compact group (or more generally a group action) is applied to demonstrate that analogues of the familiar "Hausdorff Paradox" exist if and only if the group (group action) is nonamenable. Moreover, related and stronger formulations are conjectured and discussed.

0. INTRODUCTION

In 1914 Hausdorff [5, 469–472] showed that the 2-sphere K^3 may be partitioned into four disjoint subsets A, B, C, D such that:

(1) D is countable,

(2) There is a 120° rotation whose iterates carry A, B, C into one another,

(3) There is a 180° rotation which carries each of A, B, C onto the (disjoint) union of the other two.

This implies that there can be no finite, finitely additive (nonnegative) measure on *all* subsets of K^3 which is invariant under *all* rotations because it would then follow that each of A, B, C would have to simultaneously carry both $\frac{1}{3}$ and $\frac{1}{2}$ of the total (finite) mass (since it follows without difficulty that D—or any countable set—would have to have measure zero). This decomposition of K^3, consequently called "The Hausdorff Paradox," was further refined by Banach and Tarski [1] and later considered by Von Neumann [7] who placed it in an appropriate abstract group theoretic setting.

In this paper we describe when certain related "anomalies" can arise in the context of abstract topological groups and show that nonamenable groups, or more generally nonamenable group actions, are necessary and sufficient for the occurrence of an abstract analogue of the Hausdorff Paradox (HP). Finally, further stronger analogues are conjectured and discussed.

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The idea of formulating analogues of (HP) in an abstract setting and its relationship to amenability was brought to this author's attention by Sherman [6] in which he considers this question in the case of *discrete* groups only. His arguments have negligible overlap with those which follow, and moreover lead to weaker "paradoxes" than our (as is immediately seen upon interpreting them in the language of "multisets" as developed in the following section I). Sherman does not use the notion of a "multiset"; instead he works with certain cyclic extensions of the group in question and his "paradoxes" are stated in this context. More importantly, his presentation involves an intricate (and very lengthy) functional extension argument which does not arise in our development (e.g., see our proof of Theorem 2.3 where we show how paradoxes arise directly upon appealing to the well known criterion (D) for (non) amenability).

I. Notation and Some Generalities

In the following G will be a fixed locally compact topological group, and any subset of G considered or under discussion is *always* assumed (or proved) to belong to the σ algebra of all Borel sets of G (generated by all the closed sets in the given topology). In general we follow the notation and terminology of Greenleaf's monograph [4]. We also need the notion of a "multiset" of G, and the remainder of this section is concerned with the definition and development of their basic properties as needed in the sequel.

Definition 1.1. (a) A multiset χ of G is a (Borel measurable) simple function

$$\chi\colon G \to Z^+ \doteqdot \{0, 1, 2,\dots\}.$$

The family of all multisets of G is denoted by $\mathscr{M} = \mathscr{M}(G)$.

(b) The base of a multiset χ in $\mathscr{M}$, $B(\chi)$, is the subset of G defined by

$$B(\chi) \doteqdot \{g \text{ in } G\colon \chi(g) > 0\},$$

i.e. what may be referred to as the "strict" support of χ rather than the "topological" support of χ which is usually taken to be closed.

(c) The height of χ at g in G is the integer $\chi(g)$.

(d) The height of χ, $\|\chi\|$, is defined by

$$\|\chi\| = \max\{\chi(g)\colon g \text{ in } G\}.$$

Comments. 1. In the following we shall develop the theory of multisets in a "geometric" framework and this accounts for our choice of words above.

2. It is natural to identify a subset E of G with the multiset χ of height one and base E, i.e. its characteristic function χ_E, and conversely any multiset χ with $\|\chi\| = 1$ may be identified with the subset $B(\chi)$ of G.

3. Note that $\mathscr{M}$ is a set of functions closed under addition, multiplication by nonnegative integers, (left) translation by elements of G, and moreover has a natural order and lattice structure. This simple observation will be appealed to shortly.

We now turn to the "geometric" interpretation of $\mathscr{M}$ and make the following central definition:

DEFINITION 1.2. (i) $\mathscr{M}^*$ is the family of all formal expressions $S = \bigcup \{n_i E_i \colon 1 \leqslant i \leqslant k\}$ where k is any positive integer, and E_i is a subset of G and n_i is in N for $1 \leqslant i \leqslant k$.

(ii) To each S in $\mathscr{M}^*$ correspond the function χ_S on G defined by $\chi_S \doteq \sum \{n_i \chi_{E_i} \colon 1 \leqslant i \leqslant k\}$, where χ_E denotes the characteristic function of the subset E.

(iii) For S and T in $\mathscr{M}^*$ write $S \equiv T$ iff $\chi_S = \chi_T$ (as functions on G).

As a simple consequence of Definition 1.2, we have:

PROPOSITION 1.3. (i) $\mathscr{M} = \{\chi_S \colon S \text{ in } \mathscr{M}^*\}$.

(ii) The relation $\equiv$ *on* $\mathscr{M}^*$ *is an equivalence relation and the family* $[\mathscr{M}^*]$ *of equivalence classes* $[S]$ *of* $\mathscr{M}^*$ *obtained is in a* $1 \leftrightarrow 1$ *correspondence with* $\mathscr{M}$ *via* $[S] \leftrightarrow \chi_S$.

Comment. Reordering the terms $n_i E_i$ in the expression for S does not affect the equivalence class $[S]$ and consequently when working in $[\mathscr{M}^*]$—as is usually the case— we treat the "union" as unordered, i.e. it depends only on the $n_i E_i$ occurring and not on their particular ordering in S.

DEFINITION 1.4. S in $\mathscr{M}^*$ is said to be

(i) a geometric representation of χ in $\mathscr{M}$ iff $\chi = \chi_S$,

(ii) purely geometric iff each $n_i = 1$ in the expression for S,

(iii) disjunctive iff $E_i = E_j$ or $E_i \cap E_j = \varnothing$ for all $1 \leqslant i, j \leqslant k$ in the expression for S,

(iv) totally disjunctive iff $E_i \cap E_j = \varnothing$ for all $1 \leqslant i, j \leqslant k$ and $i \neq j$ in the expression for S.

Comments. (1) For a subset E of G the notation χ_E traditionally represents the characteristic function of E which is consistent with our notation (upon identifying E with $\bigcup \{E \colon 1 \leqslant i \leqslant 1\}$).

(2) Associated with any χ in $\mathscr{M}$ there are two canonic geometric representations in $\mathscr{M}^*$:

(a) $\bigcup\{E^i: 1 \leqslant i \leqslant \|\chi\|\}$, where $E^i \doteq \{g \text{ in } G: \chi(g) \geqslant i\}$;

(b) $\bigcup\{iE_i: 1 \leqslant i \leqslant \|\chi\|\}$, where $E_i \doteq \{g \text{ in } G: \chi(g) = i\}$.

Note that the representation in (a) is purely geometric whereas that in (b) is totally disjunctive.

(3) Each S in $\mathscr{M}^*$ for which $\chi_S = \chi$ gives a geometric model for the multiset χ as follows: imagine G to lie along a horizontal line and "above" each subset E_i place a "rectangle" of "height" n_i (and "base" projecting down to E_i) for $1 \leqslant i \leqslant k$ in such a fashion that these rectangles do not "overlap." Then the (disjoint) union of these k "rectangles" in the "plane" is a geometric model for χ. Note that the base of χ, $B(\chi)$, is simply that portion of G which lies below the "rectangular union" described above and that the height of χ at g is the sum of the "heights" of all "rectangles" above g (in *any* geometric model S for χ). Finally if S is disjunctive, in the corresponding geometric model the "rectangles" may be "stacked" without overlap or gaps and in this case the model is particularly simple. The two canonic representations of comment (2) should be viewed in this fashion to aid intuition.

We now interpret the notions on $\mathscr{M}$ (Comment (3) after Definition 1.1) geometrically in $\mathscr{M}^*$:

DEFINITION 1.5. (1) For $S = \bigcup\{n_iE_i: 1 \leqslant i \leqslant k\}$ and $T = \bigcup\{m_jF_j: 1 \leqslant j \leqslant r\}$ in $\mathscr{M}^*$ and n in N and g in G:

(i) $nS \doteq \bigcup\{(n\, n_i)E_i: 1 \leqslant i \leqslant k\}$,

(ii) $gS \doteq \bigcup\{n_i(gE_i): 1 \leqslant i \leqslant k\}$ where $gE = \{gh: h \text{ in } E\}$,

(iii) $S \cup T \doteq \bigcup\{w_\nu H_\nu: 1 \leqslant \nu \leqslant k + r\}$ where $w_\nu = n_\nu$ and $H_\nu = E_\nu$ for $1 \leqslant \nu \leqslant k$ and $w_\nu = m_{\nu-k}$ and $H_\nu = F_{\nu-k}$ for $k < \nu \leqslant k + r$.

(2) For S and T in $\mathscr{M}^*$, write $S \subseteq T$ iff $\chi_S \leqslant \chi_T$.

Comment. It follows from (1 iii) (using any sequence of associations) that $(n_1E_1) \cup (n_2E_2) \cup \cdots \cup (n_kE_k) = \bigcup\{n_iE_i: 1 \leqslant i \leqslant k\}$. Consequently, the "formal" expressions of $\mathscr{M}^*$ in fact behave as "unions with multiplicity" as further clarified in the following:

PROPOSITION 1.6. (1) *The notions of Definition* 1.5 *are in fact well defined on* $[\mathscr{M}^*]$. *More precisely*:

(i) $S \equiv \hat{S}$ implies $nS \equiv n\hat{S}$ *for n in N, and* $gS \equiv g\hat{S}$ *for g in G.*

(ii) $S \equiv \hat{S}$ *and* $T \equiv \hat{T}$ *implies* $S \cup T \equiv \hat{S} \cup \hat{T}$ *and* $S \subseteq T$ *iff* $\hat{S} \subseteq \hat{T}$.

(2) *The above operations when considered on* $[\mathscr{M}^*]$ *are identical/compatible with the operations on* $\mathscr{M}$ *carried to* $[\mathscr{M}^*]$ *via the canonic* $1 \leftrightarrow 1$ *correspondence* $\chi_S \leftrightarrow [S]$. *More precisely*:

(i) *$\chi_S \leftrightarrow [S]$ implies $n\chi_S \leftrightarrow [nS]$ and ${}_g\chi_S \leftrightarrow [gS]$ (recall $({}_gf)(x) \doteq f(g^{-1}x)$ for any function f on G).*

(ii) *$\chi_S \leftrightarrow [S]$ and $\chi_T \leftrightarrow [T]$ implies $\chi_S + \chi_T \leftrightarrow [S \cup T]$.*

(3) *$[\mathscr{M}^*]$ is a commutative semigroup with identity $[\varnothing]$ under the operation $\bigcup$. Moreover for S and T in $\mathscr{M}^*$, m and n in N, and g and h in G:*

(i) $(n+m)S \equiv (nS) \cup (mS)$,

(ii) $n(mS) \equiv (nm)S$,

(iii) $n(S \cup T) \equiv (nS) \cup (nT)$,

(iv) $g(hS) \equiv (gh)S$,

(v) $g(nS) \equiv n(gS)$,

(vi) *$S \subseteq \bar{S}$ and $T \subseteq \bar{T}$ implies $S \cup T \subseteq \bar{S} \cup \bar{T}$ and $nS \subseteq n\bar{S}$, $gS \subseteq g\bar{S}$.*

(4) *If $S \subseteq \hat{S}$ in $\mathscr{M}^*$, there is a unique class $\mathscr{C}$ of $[\mathscr{M}^*]$ such that $S \cup C \equiv \hat{S}$ for all C in $\mathscr{C}$ (this may be considered to be the multiset "difference").*

Proof. The following facts are easily verified for any $S, T \in \mathscr{M}$, $n \in N$, $g \in G$:

$$\chi_{nS} = n\chi_S, \qquad \chi_{gS} = {}_g\chi_S, \qquad \chi_{S \cup T} = \chi_S + \chi_T;$$

from these, (i) and (ii) follow directly. Statement (2) also follows.

Next, (3) follows from (2) since the corresponding properties are trivially valid for $\mathscr{M}$. Finally (4) is true since $\chi_{\hat{S}} - \chi_S \geqslant 0$ is in $\mathscr{M}$ and consequently $S \cup C \equiv \hat{S}$ iff $\chi_S + \chi_C = \chi_{\hat{S}}$, i.e. $\chi_C = \chi_{\hat{S}} - \chi_S$ or equivalently iff C is in the class $\mathscr{C}$ of $[\mathscr{M}^*]$ which corresponds to $\chi_{\hat{S}} - \chi_S$ in $\mathscr{M}$ under the canonic correspondence between $\mathscr{M}$ and $[\mathscr{M}^*]$.

Definition 1.7. If $S, \hat{S}$ in $\mathscr{M}^*$, and $S \equiv \hat{S}$ where $S = \bigcup \{n_i E_i : 1 \leqslant i \leqslant k\}$, $\hat{S} = \bigcup \{\hat{n}_j \hat{E}_j : 1 \leqslant j \leqslant \hat{k}\}$, we say $\hat{S}$ is a refinement of S and write $\hat{S} > S$ iff there is a refinement map $\omega: \{1, 2,..., \hat{k}\} \to \{1, 2,..., k\}$ such that $n_i E_i \equiv \bigcup \{\hat{n}_j \hat{E}_j : j \text{ in } \omega^{-1}(i)\}$ for $1 \leqslant i \leqslant k$. Moreover, if $S_\nu \equiv \hat{S}$ and $\hat{S} > S_\nu$ for $1 \leqslant \nu \leqslant m$ we then say $\hat{S}$ is a common refinement of the S_ν, $1 \leqslant \nu \leqslant m$.

Comment. The function ω in Definition 1.7 need not be unique—if one exists at all—as *is* the case with a refinement of a partition of a subset, i.e., a refinement of a multiset of height one. This absence of unicity makes many of our arguments "noncanonic" and somewhat involved, but these technical problems are readily surmounted. Finally, note that any refinement of a purely geometric representation is again of the same type whereas the same is not true of a general disjunctive representation.

Proposition 1.8. (i) *$\hat{S} > S$ and $\hat{\hat{S}} > \hat{S}$ implies $\hat{\hat{S}} > S$.*

(ii) *Given $S_1, ..., S_m$ in $\mathscr{M}^*$ with $S_i \equiv S_j$ for $1 \leqslant i, j \leqslant m$ there is a common refinement $\hat{S}$ which is, moreover, purely geometric and disjunctive.*

Proof. (i) The transitivity asserted follows immediately by taking the composition of the two index maps involved (any two "acceptable" index maps may be chosen in establishing the transitivity).

To prove (ii), assume $S_\nu = \bigcup \{n_i^\nu E_i^\nu: 1 \leqslant i \leqslant k_\nu\}$ for $1 \leqslant \nu \leqslant m$. Arrange all the subsets E_i^ν, $1 \leqslant \nu \leqslant m$ and $1 \leqslant i \leqslant k_\nu$, in some sequential order $E_1, \ldots, E_d$ and consider the at most 2^d nonempty subsets E^i of the form $F_1 \cap F_2 \cap \cdots \cap F_d$ where each F_i is either E_i or $E_i' = G - E_i$. Then, as is well known, these sets E^i are disjoint and each E_i is the union of at most 2^{d-1} of them (all the (nonempty) ones for which $F_i = E_i$). Consequently, each $n_i^\nu E_i^\nu$ is the multiset union of at most 2^{d-1} of the E^i each repeated n_i^ν times. Taking $\nu = 1$ (for example) and writing each $n_i^1 E_i^1$, $1 \leqslant i \leqslant k_1$, as a multiset union of various E^i with repetitions and then taking the multiset union of these k_1 multiset unions we finally obtain a refinement $\hat{S}$ of S_1 which is purely geometric and disjunctive and consists of each of the 2^d sets E^i repeated with varying multiplicities. By the same reasoning each of the S_ν with $\nu > 1$ also has a refinement of the same form, i.e. being a multiset union of the E^i repeated with varying multiplicities. But since the E^i are disjoint all these refinements for each ν are in fact identical (up to the order of the summands) and consequently the refinement $\hat{S}$ of S_1 is in fact a common refinement of all the S_ν, $1 \leqslant \nu \leqslant m$.

We conclude this section with the basic concept of this paper and a development of some of its properties:

DEFINITION 1.9. χ_1 and χ_2 in $\mathscr{M}$ are said to be congruent written $\chi_1 \cong \chi_2$, iff there are purely geometric S and T in $\mathscr{M}^*$ such that $\chi_1 = \chi_S$ and $\chi_2 = \chi_T$ and $S = \bigcup \{E_i: 1 \leqslant i \leqslant k\}$, $T = \bigcup \{g_i E_i: 1 \leqslant i \leqslant k\}$ for appropriate g_i in G and subsets E_i of G, $1 \leqslant i \leqslant k$. Moreover, S and T in $\mathscr{M}^*$ are said to be congruent, written $S \cong T$, iff $\chi_S \cong \chi_T$.

Comment. There is in fact no loss of generality in requiring the S and T in $\mathscr{M}^*$ which establish the congruence to be ***purely*** geometric as any refinement of S may also be used in establishing the congruence. Of course any S in $\mathscr{M}^*$ has a natural purely geometric refinement obtained by refining

$$nE \equiv (\overbrace{E \cup \cdots \cup E}^{n \text{ times}}).$$

PROPOSITION 1.10. 1. *The congruence relationship $\cong$ is an equivalence relation on both $\mathscr{M}$ and $\mathscr{M}^*$.*

2. *Upon denoting the congruence classes of $\mathscr{M}$ by $(\mathscr{M})$ and those of $\mathscr{M}^*$ by $(\mathscr{M}^*)$ we have:*

(i) $(S) \leftrightarrow (\chi_S)$ *is a* $1 \leftrightarrow 1$ *correspondence between* $(\mathscr{M}^*)$ *and* $(\mathscr{M})$,

(ii) $[S] \subseteq (S)$ *for any* S *in* $\mathscr{M}^*$;

3. *$(\mathscr{M}^*)$ inherits the operations on $\mathscr{M}^*$ (see Definitions 1.5.1). More precisely:*

(a) *$S \cong T$ implies $nS \cong nT$ and $gS \cong gT$ $(\cong S)$ for all n in N and g in G,*

(b) *$S_1 \cong T_1$ and $S_2 \cong T_2$ implies $S_1 \cup S_2 \cong T_1 \cup T_2$ (and more generally for any finite union).*

That is, $n(S) \doteq (nS)$, $g(S) \doteq (gS)$, $(S) \cup (T) \doteq (S \cup T)$ are well defined on $(\mathscr{M}^*)$ (of course, the action of G is trivial).

Comment. The relation of containment $(\subseteq)$ on $\mathscr{M}^*$ of course does not transfer in any coherent fashion to $(\mathscr{M}^*)$. This is precisely the source of the anomalies in the next section.

Proof. (1) We verify that $\cong$ is an equivalence on $\mathscr{M}$ only as it immediately follows from the definition of $\cong$ on $\mathscr{M}^*$ that it must also then be an equivalence on $\mathscr{M}^*$. Reflexivity is trivial (all $g_i = e$), and symmetry is immediate since $S = \bigcup \{E_i\colon 1 \leqslant i \leqslant k\}$ and $T = \bigcup \{g_i E_i\colon 1 \leqslant i \leqslant k\}$ and $\chi_1 = \chi_S$, $\chi_2 = \chi_T$ implies $\chi_1 = \chi_{\hat{S}}$ and $\chi_2 = \chi_{\hat{T}}$ where $\hat{S} = \bigcup \{(g_i^{-1})\hat{E}_i\colon 1 \leqslant i \leqslant k\}$, $\hat{T} = \bigcup \{\hat{E}_i\colon 1 \leqslant i \leqslant k\}$ and $\hat{E}_i \doteq g_i E_i$. To verify transitivity assume $\chi_1 \cong \chi_2$ and $\chi_2 \cong \chi_3$ and $\chi_1 = \chi_S$, $\chi_2 = \chi_T$, $\chi_2 = \chi_{S^*}$, $\chi_3 = \chi_{T^*}$ where $S = \bigcup \{E_i\colon 1 \leqslant i \leqslant k\}$, $T = \bigcup \{g_i E_i\colon 1 \leqslant i \leqslant k\}$, $S^* = \bigcup \{E_i^*\colon 1 \leqslant i \leqslant r\}$, $T^* = \bigcup \{g_i^* E_i^*\colon 1 \leqslant i \leqslant r\}$. Since $\chi_T = \chi_2 = \chi_{S^*}$ we have $T \equiv S^*$ and consequently by Proposition 1.8 (ii) there is a common (purely geometric) refinement $\hat{S} = \bigcup \{\hat{E}_j\colon 1 \leqslant i \leqslant \hat{k}\}$. Let $\omega_1\colon \{1,..., \hat{k}\} \to \{1,..., k\}$ and $\omega_2\colon \{1,..., \hat{k}\} \to \{1,..., r\}$ be such that $g_i E_i \equiv \bigcup \{\hat{E}_j\colon j \text{ in } \omega_1^{-1}(i)\}$, $1 \leqslant i \leqslant k$ and $E_i^* \equiv \bigcup \{\hat{E}_j\colon j \text{ in } \omega_2^{-1}(i)\}$, $1 \leqslant i \leqslant r$ (as noted before, ω_1 and ω_2 need not be unique). Consequently $E_i \equiv \bigcup \{g_1^{-1}\hat{E}_j\colon j \text{ in } \omega_1^{-1}(i)\}$, $1 \leqslant i \leqslant k$, and $g_i^* E_i^* \equiv \bigcup \{g_i^* \hat{E}_j\colon j \text{ in } \omega_2^{-1}(i)\}$, $1 \leqslant i \leqslant r$. But this implies

$$\begin{aligned} S &= \bigcup \{E_i\colon 1 \leqslant i \leqslant k\} \equiv \bigcup \{g_i^{-1}\hat{E}_j\colon 1 \leqslant i \leqslant k \text{ and } j \text{ in } \omega_1^{-1}(i)\} \\ &= \bigcup \{g_{\omega_1(j)}^{-1}\hat{E}_j\colon 1 \leqslant j \leqslant \hat{k}\} \text{ and } T^* = \bigcup \{g_i^* E_i^*\colon 1 \leqslant i \leqslant r\} \\ &\equiv \bigcup \{g_i^* \hat{E}_j\colon 1 \leqslant i \leqslant r \text{ and } j \text{ in } \omega_2^{-1}(i)\} = \bigcup \{g_{\omega_2(j)}^* \hat{E}_j\colon 1 \leqslant j \leqslant \hat{k}\}. \end{aligned}$$

Finally, let $\tilde{S} \doteq \bigcup \{F_j\colon 1 \leqslant j \leqslant \hat{k}\}$ where $F_j \doteq g_{\omega_1(j)}^{-1}\hat{E}_j$ for $1 \leqslant j \leqslant \hat{k}$, and $\tilde{T} \doteq \bigcup \{\tilde{g}_j F_j\colon 1 \leqslant j \leqslant \hat{k}\}$ where $\tilde{g}_j \doteq g_{\omega_2(j)}^* g_{\omega_1(j)}$ for $1 \leqslant j \leqslant \hat{k}$. Then what we have shown directly above is $S \equiv \tilde{S}$ and $T^* \equiv \tilde{T}$, and we finally have $\chi_1 = \chi_S = \chi_{\tilde{S}}$ and $\chi_3 = \chi_{T^*} = \chi_{\tilde{T}}$ and therefore by Definition 1.9 $\chi_1 \cong \chi_3$.

Statement 2 follows immediately from Definition 1.9 while the assertions of 3 are trivialities.

PROPOSITION 1.11. (1) *If S, T_1, T_2 in $\mathscr{M}^*$ are such that $S \cong T_1 \cup T_2$ there exist S_1 and S_2 in $\mathscr{M}^*$ such that $S_1 \cong T_1$, $S_2 \cong T_2$ and $S \equiv S_1 \cup S_2$.*

(2) *If S, T, T_1 in $\mathscr{M}^*$ are such that $S \cong T$ and $T_1 \subseteq T$ there exists S_1 in $\mathscr{M}^*$ such that $S_1 \subseteq S$ and $S_1 \cong T_1$.*

Comment. Moreover, the following generalization of (1) holds but we do not prove it here as it is not needed in the sequel: If S_i, $1 \leqslant i \leqslant m$, and T_j, $1 \leqslant j \leqslant n$ in $\mathscr{M}^*$ satisfy

$$\bigcup \{S_i : 1 \leqslant i \leqslant m\} \cong \bigcup \{T_j : 1 \leqslant j \leqslant n\}$$

then there exist $S_{i,j}$ in $\mathscr{M}^*$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, such that $S_i \equiv \bigcup \{S_{i,j}: 1 \leqslant j \leqslant n\}$ for $1 \leqslant i \leqslant m$ and $T_j \cong \bigcup \{S_{i,j}: 1 \leqslant i \leqslant m\}$ for $1 \leqslant j \leqslant n$.

Proof. (1) Since $S \cong T_1 \cup T_2$ we have for appropriate g_i in G and subsets E_i of G, $1 \leqslant i \leqslant k$: $S \equiv \bigcup \{E_i : 1 \leqslant i \leqslant k\} \doteqdot S^*$, $T_1 \cup T_2 \equiv \bigcup \{g_i E_i : 1 \leqslant i \leqslant k\} \doteqdot T^*$. Moreover, assume $T_\nu = \bigcup \{n_i^{(\nu)} F_i^{(\nu)} : 1 \leqslant i \leqslant k_\nu\}$ for $\nu = 1, 2$ and thus $T_1 \cup T_2 = \bigcup \{n_i F_i : 1 \leqslant i \leqslant k_1 + k_2\}$ where $n_i F_i = n_i^{(1)} F_i^{(1)}$ for $1 \leqslant i \leqslant k_1$ and $n_i F_i = n_{i-k_1}^{(2)} F_{i-k_1}^{(2)}$ for $k_1 < i \leqslant k_1 + k_2$. Upon taking a common (purely geometric) refinement of $T_1 \cup T_2$ and T^*, say $\bigcup \{\hat{E}_j : 1 \leqslant j \leqslant \hat{k}\}$, we construct S_1 and S_2 as follows: Assume $\omega_1 : \{1,..., \hat{k}\} \to \{1,..., k\}$ and $\omega_2 : \{1,..., \hat{k}\} \to \{1,..., k_1 + k_2\}$ are refinement maps for T^* and $T_1 \cup T_2$ respectively, and let $J_\nu \subseteq \{1,..., \hat{k}\}$, $\nu = 1, 2$, be defined by $J_1 \doteqdot \omega_2^{-1}(\{1,..., k_1\})$, $J_2 \doteqdot \omega_2^{-1}\{(k_1 + 1,..., k_1 + k_2\})$. Then $T_\nu \equiv \bigcup \{\hat{E}_j : j \text{ in } J_\nu\}$ for $\nu = 1, 2$. Next, for each j, $1 \leqslant j \leqslant \hat{k}$, define the subset F_j by $g_{\omega_1(j)}^{-1} \hat{E}_j$ and note that

$$\begin{aligned}
&\bigcup \{F_j : 1 \leqslant j \leqslant \hat{k}\} \\
&\quad = \bigcup \{g_{\omega_1(j)}^{-1} \hat{E}_j : 1 \leqslant j \leqslant \hat{k}\} \\
&\quad \equiv \bigcup \{g_i^{-1} \hat{E}_j : 1 \leqslant i \leqslant k \text{ and } \omega_1(j) = i\} \\
&\quad \equiv \bigcup \{g_i^{-1} \cup \{\hat{E}_j : j \text{ in } \omega_1^{-1}(i)\} : 1 \leqslant i \leqslant k\} \\
&\quad \equiv \bigcup \{g_i^{-1}(g_i E_i) : 1 \leqslant i \leqslant k\} = \bigcup \{E_i : 1 \leqslant i \leqslant k\} = S^* \equiv S.
\end{aligned}$$

Consequently if $S_\nu \doteqdot \bigcup \{F_j : j \text{ in } J_\nu\}$ for $\nu = 1, 2$ we have $S_1 \cup S_2 \equiv S$. Finally, since $T_\nu \equiv \bigcup \{\hat{E}_j : j \text{ in } J_\nu\}$ and $S_\nu = \bigcup \{g_{\omega_1(j)}^{-1} \hat{E}_j : j \text{ in } J_\nu\}$ for $\nu = 1, 2$ we also have $T_\nu \cong S_\nu$ for $\nu = 1, 2$ and we are done.

Assertion (2) is an easy consequence of (1). For by Proposition 1.6.4 there is a T_2 in $\mathscr{M}^*$ such that $T \equiv T_1 \cup T_2$ and since $S \cong T \cong T_1 \cup T_2$ we may take S_1 to be the S_1 of assertion (1).

II. Analogues of (HP) in Abstract Groups

In this section we formulate our results in terms of the concepts of multiset and congruence developed in part I. The idea is to reduce a known characterization for the amenability of G on the function space level to a more pictorial formulation directly in terms of the geometry of G. Most certainly multisets are not totally geometric objects, rather they reside somewhere between the geometric concept of set and the analytic concept of function. Nevertheless, the results herein do bear a resemblance to (HP) and it is hoped that further research (along the lines indicated in Part III perhaps) will lead to stronger analoglies with (HP).

The following result is a simple reworking of the standard characterization of the amenability of a group G in terms of the existence of a LIM [4, 26 ff.]

PROPOSITION 2.1. *The locally compact group G is amenable iff there is a nonzero function m: $(\mathscr{M}^*) \to [0, +\infty)$ such that $m((S) \cup (T)) = m((S)) + m((T))$ for all (S) and (T) in $(\mathscr{M}^*)$.*

Proof. If G is amenable then let $\hat{m}$ be a LIM on $L_\infty(G)$, and define m on $\mathscr{M}^*$ by $m(S) \doteq \hat{m}(\chi_S)$ for S in $\mathscr{M}^*$ (or more precisely $\hat{m}(\{\chi_S\})$ where $\{\chi_S\}$ is the L_∞ class of χ_S). We now show that m is in fact well defined on $(\mathscr{M}^*)$ and additive. To this end take S and T in $\mathscr{M}^*$ with $S \simeq T$, i.e. there are $\hat{S}$ and $\hat{T}$ in $\mathscr{M}^*$ with $S \equiv \hat{S}$, $T \equiv \hat{T}$ and $\hat{S} = \bigcup \{\hat{E}_i: 1 \leqslant i \leqslant \hat{k}\}$, $\hat{T} = \bigcup \{g_i \hat{E}_i: 1 \leqslant i \leqslant \hat{k}\}$ for appropriate subsets $\hat{E}_i$ of G and g_i in G, $1 \leqslant i \leqslant \hat{k}$. Then

$$\begin{aligned} m(\hat{S}) &= \hat{m}(\chi_{\hat{S}}) = \hat{m}\left(\sum \{\chi_{\hat{E}_i}: 1 \leqslant i \leqslant \hat{k}\}\right) \\ &= \sum \{\hat{m}(\chi_{\hat{E}_i}): 1 \leqslant i \leqslant \hat{k}\} \text{ (by the additivity of } \hat{m}) \\ &= \sum \{\hat{m}({}_{g_i}\chi_{\hat{E}_i}): 1 \leqslant i \leqslant \hat{k}\} \text{ (by the left invariance of } \hat{m}) \\ &= \sum \{\hat{m}(\chi_{g_i \hat{E}_i}): 1 \leqslant i \leqslant \hat{k}\} = \hat{m}\left(\sum \{\chi_{g_i E_i}: 1 \leqslant i \leqslant \hat{k}\}\right) \\ &= \hat{m}(\chi_{\hat{T}}) = m(\hat{T}). \end{aligned}$$

Thus, since $\chi_S = \chi_{\hat{S}}$ and $\chi_T = \chi_{\hat{T}}$, $m(S) = m(T)$ and m is in fact well defined on $(\mathscr{M}^*)$. Additivity now follows since

$$\begin{aligned} m((S) \cup (T)) &= m((S \cup T)) = \hat{m}(\chi_{S \cup T}) = \hat{m}(\chi_S + \chi_T) \\ &= \hat{m}(\chi_S) + \hat{m}(\chi_T) = m(S) + m(T) = m((S)) + m((T)). \end{aligned}$$

Finally, m is nonzero since $m((G)) = \hat{m}(\chi_G) = 1$.

Conversely, if m satisfies the conditions of 2.1 we define ("abuse of notation") m on $\mathscr{M}^*$ itself by $m(S) \doteq m((S))$ and also on $\mathscr{M} \leftrightarrow [\mathscr{M}^*]$ by $m(\chi) \doteq m((S))$ where S is any member of $\mathscr{M}^*$ for which $\chi = \chi_S$ (which is well defined since $[S] \subseteq (S)$]. We now extend m to $BM(G)$, the space of all bounded Borel measurable (real) functions on G: first, for any simple function $f = \sum \{\alpha_i \chi_{E_i}: 1 \leqslant i \leqslant k\}$, we set $m(f) \doteq \sum \{\alpha_i m(E_i): 1 \leqslant i \leqslant k\}$, which is easily seen to be well defined by appealing to a common refinement as in 1.8(ii) (and moreover is consistent with the definition of m on $\mathscr{M}$). Next, since the extended functional m so defined on $\mathscr{S} = \{$all bounded (real) simple Borel measurable functions on $G\}$ is readily seen to be non-negative and sup norm continuous and $\mathscr{S}$ is sup norm dense in $BM(G)$, m consequently has a unique extension, also denoted by m, to $BM(G)$. This extended m on $BM(G)$ clearly is a non-negative functional and inherits left invariance from m on $\mathscr{M}$. Finally, note that $m_0 \doteq m(1) = m(G)$ is positive; for otherwise $m(G) = 0$ which would imply $m(nG) = 0$ for all n in N and since any S in $\mathscr{M}^*$ satisfies $S \subseteq \| G \| G$ it would follow that $m \equiv 0$ on $\mathscr{M}$ since m is monotone. Consequently the functional m/m_0 is a LIM on $BM(G)$. We are now ready to construct a LIM $\hat{m}$ on $L_\infty(G)$ by "smoothing" m/m_0. To this end let U_0 be any relatively compact open subset of G and let ϕ_0 be the normalized characteristic function of U_0. We now define $\hat{m}$ on $BM(G)$ by $\hat{m}(f) \doteq m(f * \phi_0^\sim)/m_0$, f in $BM(G)$, where $\phi_0^\sim(g) \doteq \phi_0(g^{-1})$ and consequently $(f * \phi^\sim)(g) \doteq \int_G f(gy)\, \phi(y)\, d\mu(y)$, where μ is left Haar measure on G. Next note that if f and g in $BM(G)$ are identified in L_∞, i.e. $f - g = 0$ a.e. μ, then $f * \phi_0^\sim \equiv g * \phi_0^\sim$ and thus $\hat{m}$ is in fact defined on L_∞, and since $\hat{m}$ inherits the appropriate properties from m/m_0 on BM it follows that $\hat{m}$ is a LIM on L_∞. Therefore G is amenable and the proof of the proposition is completed. Note that "smoothing" may really be necessary as, for example, there exist LIM's on $BM(T)$, $T =$ the circle, which are "supported" on Borel sets of Lebesgue measure zero.

Corollary 2.2. *If there exist S and T in $\mathscr{M}^*$ such that $S \simeq T$ and $S \cup G \subsetneq T$ then G is not amenable.*

Proof. If G is amenable let m be as in 2.1 and extended to $\mathscr{M}^*$ by $m(S) = m((S))$. Consequently m is additive and nonnegative on $\mathscr{M}^*$ and therefore also monotone. Consequently $m(S) + m(G) = m(S \cup G) \leqslant m(T)$, and since $S \simeq T$ also $m(S) = m((S)) = m((T)) = m(T)$. Therefore $m(G) \leqslant 0$ which implies $m(G) = 0$ which is impossible since this would make $m \equiv 0$ (as seen already in the proof of 2.1).

That the converse of 2.2 is also valid is a direct consequence of the Von-Neumann/Dixmier criterion (D) for the amenability of G [4, p. 25]:

> The locally compact group G is amenable iff for
> every finite family $f_1, \ldots, f_k$ in $L_\infty(G)$ and $g_1, \ldots, g_k$ in G
> $$\operatorname{ess\,inf} \sum \{{}_{g_i}f_i - f_i\colon 1 \leqslant i \leqslant k\} \leqslant 0. \tag{D}$$

Theorem 2.3. *The locally compact group G is not amenable iff there exist $f_1, \ldots, f_k$ in $L_\infty(G)$, $g_1, \ldots, g_k$ in G, and $\epsilon > 0$ such that*

$$\operatorname{ess\,inf} \sum \{{}_{g_i}f_i - f_i\colon 1 \leqslant i \leqslant k\} \geqslant \epsilon, \tag{D$'$}$$

iff there exist S and T in $\mathscr{M}^*$ such that $S \simeq T$ and $S \cup G \subsetneq T$.

Proof. The first equivalence in 2.3 is simply the contrapositive form of criterion (D). To establish the second equivalence, first assume $S \cong T$ and $S \cup G \subseteq T$. Then there are subsets E_i of G and g_i in G, $1 \leqslant i \leqslant k$, such that $\chi_S = \sum \{\chi_{E_i}: 1 \leqslant i \leqslant k\}$ and $\chi_T = \sum \{\chi_{g_i E_i}: 1 \leqslant i \leqslant k\}$. Consequently $\chi_T \geqslant \chi_{S \cup G} = \chi_S + \chi_G = \chi_S + 1$ implies

$$\sum \{\chi_{g_i E_i} - \chi_{E_i}: 1 \leqslant i \leqslant k\} = \sum \{{}_{g_i}\chi_{E_i} - \chi_{E_i}: 1 \leqslant i \leqslant k\} = \chi_T - \chi_S \geqslant 1,$$

which is an instance of (D′) (with $f_i = \chi_{E_i}$ and $\epsilon = 1$).

Conversely, assuming an instance of (D′) we may moreover assume all f_i continuous and drop the "ess inf" by smoothing/convolving any instance of (D′) on the right by an appropriate positive function ϕ ($f_i \to f_i * \phi$). Furthermore, since replacing f_i by $f_i + c_i$ for any constants c_i does not change $\sum \{{}_{g_i}f_i - f_i: 1 \leqslant i \leqslant k\}$ we may also assume all $f_i > 0$ (e.g. take $c_i > \sup |f_i|$). Next, since simple functions with rational values are dense in L_∞ we may assume each f_i is also of this form (upon decreasing ϵ slightly to $\epsilon' > 0$ and approximating the given f_i sufficiently closely in sup norm by rational valued simple functions if necessary). Finally, upon multiplying through such an instance of (D′) by any positive integer M which is a common multiple of the rational values assumed by all the f_i we obtain

$$\sum \{{}_{g_i}(Mf_i) - (Mf_i): 1 \leqslant i \leqslant k\} \geqslant M\epsilon' > 0. \tag{D''}$$

Now each of the Mf_i, $1 \leqslant i \leqslant k$, is a simple function taking on only positive integral values, i.e. Mf_i is in $\mathscr{M}$. Choose any purely geometric S_i in $\mathscr{M}^*$ such that $\chi_{S_i} = Mf_i$, $1 \leqslant i \leqslant k$. Also since the left side of (D′) above is always integral we may replace $M\epsilon'$ on the right side by 1 and obtain $\sum \{{}_{g_i}\chi_{S_i} - \chi_{S_i}: 1 \leqslant i \leqslant k\} \geqslant 1$ or equivalently

$$\sum \{\chi_{g_i S_i}: 1 \leqslant i \leqslant k\} \geqslant \sum \{\chi_{S_i}: 1 \leqslant i \leqslant k\} + 1.$$

Reinterpreting the above relation in $\mathscr{M}^*$ we have

$$\bigcup \{g_i S_i: 1 \leqslant i \leqslant k\} \supseteq \bigcup \{S_i: 1 \leqslant i \leqslant k\} \cup G.$$

Upon letting $S = \bigcup \{S_i: 1 \leqslant i \leqslant k\}$, $T = \bigcup \{g_i S_i: 1 \leqslant i \leqslant k\}$ one readily verifies that $S \cong T$ (simply write out each S_i as a multiset union of subsets of G) and as we have just shown $T \supseteq S \cup G$ and we are done.

Comment. Theorem 2.3 shows that G is not amenable iff there are congruent S and T in $\mathscr{M}^*$ such that $T \supseteq S \cup G$, and this is our first primitive analogue of (HP). It essentially shows the incompatability of the congruence concept with the notion of order/inclusion in nonamenable groups. It is stated in the terminology of multisets whereas a purely geometric formulation in terms of subsets of G would admittedly be preferable. We now carry out a portion of this program of obtaining an anomaly directly in terms of the subset geometry of G and in part III we conjecture more satisfactory analogues of (HP) to be equivalent to nonamenability.

LEMMA 2.4. *If S, T, D in $\mathscr{M}^*$ and $S \cong T$, $S \cup D \subseteq T$ then for any n in N there is a T_n in $\mathscr{M}^*$ such that $S \cong T_n$ and $S \cup nD \subseteq T_n$.*

Proof. Since $S \cup D \subseteq T$ we have $T \equiv S \cup D'$ for some D' in $\mathscr{M}^*$ with $D \subseteq D'$. Consequently by repeated application of 1.10.3b we have $S \cong T \equiv S \cup D' \cong T \cup D' \equiv (S \cup D') \cup D' \equiv S \cup 2D' \cong T \cup (2D') \equiv (S \cup D') \cup (2D') \equiv S \cup (3D')$, and by induction $S \cong S \cup (nD')$ for any n in N. The lemma follows upon taking $T_n \doteqdot S \cup nD'$ $(\supseteq S \cup nD)$.

THEOREM 2.5. *The locally compact group G is nonamenable iff*

$$\begin{aligned}&\textit{There is an integer } n_0 = n_0(G) \textit{ such that for any } n \textit{ in } N\\ &\textit{there is } S_n \textit{ in } \mathscr{M}^*,\ S_n \cong n_0 G, \textit{ and } nG \subseteq S_n .\end{aligned} \tag{H}$$

Equivalently, for any n in N there is a partitioning of G into disjoint subsets, $G = \bigcup \{\hat{E}_i: 1 \leqslant i \leqslant \hat{k}\}$ (this may also be viewed as a multiset union), and elements of G, $\{g_i^{(j)}: 1 \leqslant i \leqslant \hat{k},\ 1 \leqslant j \leqslant n_0\}$, *such that every point of G lies in at least n of the subsets $\{g_i^{(j)}\hat{E}_i: 1 \leqslant i \leqslant \hat{k},\ 1 \leqslant j \leqslant n_0\}$. (i.e. upon letting $G^{(j)} \doteqdot \bigcup \{g_i^{(j)}\hat{E}_i: 1 \leqslant i \leqslant \hat{k}\}$ (multiset union) for $1 \leqslant j \leqslant n_0$, then each $G^{(j)} \cong G$ and these n_0 multisets $G^{(j)}$—all congruent to G—amongst them "cover" G (at least) n times).*

Proof. If G is amenable the validity of (H) would clearly violate Proposition 2.1 since $m(S_n) = n_0 m(G)$ and $nm(G) \leqslant m(S_n)$ clearly implies $m(G) = 0$ if $n > n_0$. Conversely, if G is not amenable let S and T in $\mathscr{M}^*$ be as in Theorem 2.3, and set $n_0 \doteqdot \| S \|$. Then by Lemma 2.4 (with $D = G$) for each n in N there is a T_n in $\mathscr{M}^*$ such that $T_n \cong S$ and $S \cup nG \subseteq T_n$. Now let $D \equiv n_0 G - S$ in $\mathscr{M}^*$, and observe that $T_n \cup D \cong S \cup D \equiv n_0 G$. Furthermore we have $T_n \cup D \supseteq T_n \supseteq S \cup nG \supseteq nG$, and consequently $S_n \doteqdot T_n \cup D$ satisfies (H) (where $n_0 = \| S \|$). To obtain the equivalent formulation as described, in a given congruence between $n_0 G$ and S_n assume the geometric representation of $n_0 G$ used is $n_0 G = S = \bigcup \{E_i: 1 \leqslant i \leqslant k\}$. Then (by 1.8 (ii)) let $S^* = \bigcup \{E_i^*: 1 \leqslant i \leqslant k^*\}$ be a purely geometric disjunctive refinement of S. It follows that if $\hat{E}_1, \ldots, \hat{E}_{\hat{k}}$ is a complete list of all the distinct sets among the E_i^*, $1 \leqslant i \leqslant k^*$, then each occurs exactly n_0 times as an E_i^*. Moreover, if $S_n \equiv \bigcup \{g_i E_i: 1 \leqslant i \leqslant k\}$ in the given congruence between $n_0 G$ and S_n and $\omega: \{1, \ldots, k^*\} \to \{1, \ldots, k\}$ is any refinement map from S^* to S it then follows that $\bigcup \{g_i^* E_i^*: 1 \leqslant i \leqslant k^*\} \cong S_n$ where $g_i^* \doteqdot g_{\omega(i)}$. The second formulation in the statement of 2.5 is now clear.

We conclude this section with a slightly stronger variant of 2.5 which follows from a more careful application of the technique used in proving 2.4:

THEOREM 2.6. *The locally compact group G is nonamenable iff*

$$\begin{aligned}&\text{There is an integer } n_0 = n_0(G) \text{ and a sequence}\\ &\{G^{(j)}: j \text{ in } N\} \text{ in } \mathscr{M}^* \text{ such that } G^{(j)} \simeq G \text{ for each } j \text{ and}\\ &\bigcup\{G^{(j)}: 1 \leqslant j \leqslant n\} \subseteq n_0G \text{ for all } n \text{ in } N,\end{aligned} \qquad (\mathrm{H}^*)$$

i.e. each point of G occurs in at most n_0 of the $G^{(j)}$, j in N, counting multiplicity.

Comment. Upon extending multiset union to infinite unions in the obvious manner we may in fact write $\bigcup\{G^{(j)}: j \text{ in } N\} \subseteq n_0G$. Note that Theorem 2.5 follows for each n in N upon setting $S_n \equiv nG \cup (n_0G - \bigcup\{G^{(j)}: 1 \leqslant j \leqslant n\}) \equiv \bigcup (G: 1 \leqslant j \leqslant n\} \cup (n_0G - \bigcup\{G^{(j)}: 1 \leqslant j \leqslant n\}) \simeq \bigcup\{G^{(j)}: 1 \leqslant j \leqslant n\} \cup (n_0G - \bigcup\{G^{(j)}: 1 \leqslant j \leqslant n\}) \equiv n_0G$.

Proof. Since (H*) is formally stronger than (H) clearly (H*) implies G nonamenable. Conversely if G is nonamenable by Theorem 2.3 there are S and T in $\mathscr{M}^*$ with $S \simeq T$ such that $S \cup G \subseteq T$. Consequently $T \equiv S \cup G^*$ for some G^* in $\mathscr{M}^*$ with $G \subseteq G^*$ and thus $S \simeq S \cup G^*$. Consequently by 1.11.1 there are S_1 and G_1 in $\mathscr{M}^*$ such that $S_1 \simeq S$ and $G_1 \simeq G^*$ and $S_1 \cup G_1 \equiv S$. Continuing, since $S_1 \simeq S \simeq S \cup G^*$ there are S_2 and G_2 in $\mathscr{M}^*$ such that $S_2 \simeq S$ and $G_2 \simeq G^*$ and $S_2 \cup G_2 \equiv S_1$. By induction for each n in N we obtain S_n and G_n in $\mathscr{M}^*$ such that $S_n \simeq S$ and $G_n \simeq G^*$ and $S_n \cup G_n \equiv S_{n-1}$ implying for each n in N, $S \equiv (G_1 \cup G_2 \cup \cdots \cup G_n) \cup S_n$. Moreover, by 1.11.2 there are $G^{(j)} \subseteq G_j \simeq G^*$ such that $G^{(j)} \simeq G$ for each j in N and consequently, if $n_0 \doteqdot \| S \|$, $n_0G \supseteq S \equiv (G_1 \cup G_2 \cup \cdots \cup G_n) \cup S_n \supseteq G_1 \cup G_2 \cup \cdots \cup G_n \supseteq G^{(1)} \cup G^{(2)} \cup \cdots \cup G^{(n)} = \bigcup\{G^{(j)}: 1 \leqslant j \leqslant n\}$ for any n in N and (H*) is verified.

Comment. Note that any $n_0 = n_0(G)$ admissible in (H) is also admissible in (H*), and conversely.

III. CONCLUDING COMMENTS AND CONJECTURES

The following natural conjecture has particularly nice consequences:

Conjecture 3.1. $n_0 = n_0(G)$ may be taken equal to 1 in both (H) and (H*).

The implication of taking $n_0 = 1$ in (H) would be that given any n in N there is a partitioning of G into disjoint subsets, $G = \bigcup\{\hat{E}_i: 1 \leqslant i \leqslant \hat{k}\}$, and elements g_i, $1 \leqslant i \leqslant \hat{k}$, such that every point of G is in at least n of the subsets $\{g_i\hat{E}_i: 1 \leqslant i \leqslant \hat{k}\}$. Equivalently (in light of Proposition 1.11), given any n in N there is a partitioning of G into disjoint subsets, $G = \bigcup\{F_i: 1 \leqslant i \leqslant r\}$ and elements $g_i^{(j)}$ in G, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n$, such that all $\{g_i^{(j)}F_i: 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n\}$ are disjoint (this follows in light of Proposition 1.11.2 and refinement, since $nG \subseteq S_n \simeq G$). Moreover, taking $n_0 = 1$ in (H*) would yield a countable sequence of partitions of G, $G = \bigcup\{F_i^{(j)}: 1 \leqslant i \leqslant r_j\}$ for j in N, and elements $g_i^{(j)}$ in G, $1 \leqslant i \leqslant r_j$ and j in N, such that all $\{g_i^{(j)}F_i^{(j)}: 1 \leqslant i \leqslant r_j$, and j in $N\}$ are disjoint, i.e. intuitively infinitely many copies of G may be partitioned and translated so as to fit disjointly in G.

Comment. Conjecture 3.1 would follow immediately from (H) or (H*) in the presence of some form of "cancellation" with respect to congruence, e.g. $2S \cong 2T$ implies $S \cong T$. The question of cancellation laws remains open and is a natural area for further investigation.

In [2] the author established a strengthened form of criterion (D)/(D′):

PROPOSITION 3.2. *The locally compact group G is not amenable iff there is an f in $UCB(G)$, $g_1, \ldots, g_k$ and $h_1, \ldots, h_k$ in G, and $\epsilon > 0$ such that*

$$\sum \{{}_{g_i}f - {}_{h_i}f: 1 \leqslant i \leqslant k\} \geqslant \epsilon. \tag{B′}$$

Arguing as in the proof of Theorem 2.3 it also follows that f may in fact be taken to be a multiset, say $f = \chi_S$. Upon translating (B′) to multiset notation we obtain (since ϵ may then be taken to be 1):

$$\bigcup\{g_iS: 1 \leqslant i \leqslant k\} \supseteq \bigcup\{h_iS: 1 \leqslant i \leqslant k\} \cup G. \tag{B′)*}$$

The question as to whether S in (B′)* may actually be taken to be a subset of G is currently open and leads to

Conjecture 3.3. The locally compact group G is not amenable iff there is a subset E of G and $g_1, \ldots, g_k$ and $h_1, \ldots, h_k$ in G such that $\bigcup\{g_iE: 1 \leqslant i \leqslant k\} \supseteq \bigcup\{h_iE: 1 \leqslant i \leqslant k\} \cup G$.

Hopefully condition (B′)/(B′)* and other formally stronger criteria for amenability/nonamenability will lead to more faithful analogues of (HP) in the abstract setting.

In conclusion it should be noted that the proofs and ideas of this paper apply virtually verbatim to the context of group (and to some extent semigroup) actions since criterion (D) has an analogue in this more general context which is formally identical to that given herein (merely being an application of the Hahn-Banach Theorem) and moreover this criterion was essentially the only result appealed to in the development of our results. One then sees that an analogue of (HP) occurs iff the group action is nonamenable as defined in [3].

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